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Nonlinear evolution equations with non-analytic dispersion relations in 2+1 dimensions: bilocal approach

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Abstract. A method is proposed of obtaining (2+1)-dimensional nonlinear equations with non-analytic dispersion relations. Bilocal formalism is shown to make it possible to represent these equations in a form close to that for their counterparts in 1+1 dimensions.

1. Introduction

Nonlinear evolution equations with non-analytic (singular) dispersion relations (SDR equations) form an important class of equations integrable by means of the inverse spectral transform. The Maxwell–Bloch equations [1] are the well known representative of this class in 1+1 dimensions. A general construction of (1+1)-dimensional SDR equations solvable via the Zakharov–Shabat spectral problem was given by Leon [2]. As was shown by Boiti *et al* in an interesting paper [3], the SDR equations in 2+1 dimensions possess a number of peculiarities, the main one being the absence of an explicit expression for the evolution linear operator $T_2 = \partial_t - W$ which enters the Lax representation. Nevertheless, this circumstance does not prevent a construction of soliton solutions by means of the Bäcklund transformations. In particular, proposed in [3] was a (2+1)-dimensional generalization of the Maxwell–Bloch equations which had a form of a rather complicated system of four equations. In our opinion, such a complexity was caused by the fact that the approach realized in [3] was primarily based on the function W given unexplicitly. In this connection, it is seemed to be reasonable to propose another way of deriving the above class of equations without making direct use of the function W.

We will consider as a primary object a spectral transform R appearing in the framework of the $\bar{\partial}$-method [4–7]. Hence, the aim of the present paper is to obtain a hierarchy of (2+1)-dimensional nonlinear equations with non-analytic dispersion relations compatible with the linear evolution of the spectral transform R. We will demonstrate that the formalism developed by Beals and Coifman [8] for holomorphic dispersion relations can be adapted naturally for equations of interest. Our consideration relies essentially on the bilocal approach initiated by Konopelchenko and Dubrovsky [9] and elaborated to a full extent by Fokas and Santini [10, 11]. It is precisely the bilocal formalism that allows us to generate in a natural manner (2+1)-dimensional analogues of many structures which work successfully in 1+1 dimensions. We will show that the form of the SDR equations in 2+1 dimensions written in bilocal variables is very close to that for equations in 1+1 dimensions. In particular, the 'squared eigenfunction' structure typical for the (1+1)-dimensional situation also takes place in 2+1 dimensions. In the process of deriving a hierarchy of equations we shall not use, as distinct from [10], an extended integral representation for the function W

(due to the lack of an explicit expression for it). In our view, the proposed way of obtaining the recursion operator follows more closely the lines of 1+1 dimensions.

In the following, we shall restrict ourselves to the consideration of the hyperbolic spectral problem. A derivation of the relevant formulae in the case of the elliptic spectral problem does not cause principal difficulties.

2. Lax representation and the $\bar\partial$-problem

As a starting point in a construction of nonlinear SDR equations, we consider a $\bar\partial$-problem on a complex plane C ($\bar\partial \equiv \partial/\partial\bar k$):

$$\begin{aligned} &\bar\partial\phi(k) = \iint \mathrm{d}l \wedge \mathrm{d}\bar l\, \phi(l)R(k,l) \qquad k, l \in C \\ &\phi(k) = 1 + \mathrm{O}(1/k) \qquad k \to \infty \quad \phi \in SL(2, C)\,. \end{aligned} \tag{1}$$

Here the matrix R (the spectral transform) is a distribution in C^2 and a time dependence is given by the following linear evolution equation:

$$\partial_t R(k,l) = R(k,l)\Omega(k) - \Omega(l)R(k,l)\,. \tag{2}$$

In the above equation $\Omega(k)$ is a matrix-valued function on C called a dispersion relation. In a general case, $\Omega(k)$ consists of a holomorphic (polynomial) part $\Omega_{\mathrm{p}}(k)$ and a non-analytic (singular) part $\Omega_{\mathrm{s}}(k)$, i.e. $\bar\partial\Omega_{\mathrm{s}} \neq 0$ in some subset of the plane C.

Let us denote the integral in (1) as $\phi(k)R_kF$, where F is an integral operator acting on the left in accordance with (1). Hence, we write (1) as

$$\bar\partial\phi(k) = \phi(k)R_kF\,. \tag{3}$$

A solution of the $\bar\partial$-problem is given by a solution of the linear integral equation

$$\begin{aligned} \phi(k) &= 1 + \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar l}{l-k}\iint \mathrm{d}m\wedge\mathrm{d}\bar m\;\phi(m)R(l,m) \\ &= 1 + \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar l}{l-k}\,(\phi(l)R_l\,F) \equiv 1 + \phi(k)R_k\,FC_k\,. \end{aligned} \tag{4}$$

The integral operator C_k acting on the left transforms the argument k of the function in front of it to l and integrates with the weight $(2\pi\mathrm{i})^{-1}(l-k)^{-1}$ on the whole complex plane. The integral operators introduced in such a way allow us to write formally a solution of the $\bar\partial$-problem (3) as

$$\phi(k) = 1\cdot(1 - R_k\,FC_k)^{-1}\,. \tag{5}$$

A similar representation for solutions of the $\bar\partial$-problem was effectively used by Beals and Coifman [8] in the case of holomorphic functions $\Omega_{\mathrm{p}}(k)$.

Let us define a pairing for matrix-valued functions on C:

$$\langle\phi,\psi\rangle = \frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}k\wedge\mathrm{d}\bar k\;\phi(k)\,\tilde\psi(k)$$

where tilde stands for transpose. With respect to this pairing we have

$$\langle \phi C_k, \psi \rangle = -\langle \phi, \psi C_k \rangle \qquad \langle \phi R_k F, \psi \rangle = \langle \phi, \psi \hat{R}_k F \rangle \tag{6}$$

where $\hat{R}(k,l) = \tilde{R}(l,k)$. Assume, then, a parametric dependence of $R(k,l)$ on spatial variables (x, y) of the form

$$\partial_x R(k,l) = \mathrm{i} l \sigma_3 R(k,l) - \mathrm{i} k R(k,l) \sigma_3 \qquad \partial_y R(k,l) = \mathrm{i}(k-l) R(k,l) . \tag{7}$$

Taking as a basis the representation (5), it is shown in appendix A that the choice (7) is equivalent to setting the two-dimensional Zakharov–Shabat spectral problem

$$T_1 \phi \equiv (\partial_x + \sigma_3 \partial_y + Q)\phi - \mathrm{i}k[\sigma_3, \phi] = 0 \tag{8}$$

where a potential Q is defined as

$$Q(x,y) = -\mathrm{i}[\sigma_3, \langle \phi R_k F \rangle] \tag{9}$$

and we denote $\langle f, 1 \rangle \equiv \langle f \rangle$.

Now we turn to an evolution linear problem $\partial_t \phi = W\phi + \phi\,\Omega$. It follows from (5) and (2) that

$$\begin{aligned} \phi_t &= \phi\, \partial_t R_k\, F C_k (1 - R_k\, F C_k)^{-1} \\ &= (\phi R_k\, F \Omega\, C_k - \phi\, \Omega R_k\, F C_k)(1 - R_k\, F C_k)^{-1} \\ &= (\phi R_k\, F \Omega\, C_k - \phi\, \Omega)(1 - R_k\, F C_k)^{-1} + \phi\, \Omega \end{aligned}$$

which gives

$$W\phi = (\phi\, \Omega R_k\, F C_k - \phi\, \Omega)(1 - R_k\, F C_k)^{-1} . \tag{10}$$

It is shown in appendix B that (10) is reduced for $\Omega_{\mathrm{p}} = 0$ to

$$W(k)\phi(k) = -\phi(k) \bar{\partial}\, \Omega_{\mathrm{s}}(k) C_k (1 - R_k\, F C_k)^{-1} .$$

Multiplying this relation on the right by $(1 - R_k\, F C_k)$ and applying the $\bar{\partial}$-operator, we obtain

$$\bar{\partial} W\phi + W(\phi R_k\, F) - W\phi R_k\, F = -\phi\, \bar{\partial}\, \Omega_{\mathrm{s}}$$

which gives the integral equation for the function W [3]:

$$\bar{\partial} W(k) = -\phi\, \bar{\partial}\, \Omega_{\mathrm{s}} \phi^{-1}(k) + \iint \mathrm{d}l \wedge \mathrm{d}\bar{l}\ [W(l) - W(k)]\, \phi(l) R(k,l) \phi^{-1}(k) . \tag{11}$$

Hence, the function W is known only to within a solution of the integral equation (11). Nevertheless, Boiti *et al* have shown [3] that it is possible to derive SDR equations from the corresponding Lax representation with the operators T_1 and $T_2 = \partial_t - W$.

It should be noted here that (11) includes the inverse function ϕ^{-1}. However, there is not, in 2+1 dimensions (contrary to 1+1), a simple equation (like (8)) for ϕ^{-1}. Hence, a problem arises of finding a natural (2+1)-dimensional analogue of the inverse function in 1+1 dimensions. We shall show in the following section that such a function does exist and it permits the simplification of the SDR equations in [3].

3. Hierarchy and recursion operator

Let us calculate the evolution of the potential Q which is given explicitly by (9):

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \partial_t(\phi R_k F)\rangle].$$

The right-hand side can be transformed as follows:

$$\partial_t(\phi R_k F) = \partial_t \phi R_k F + \phi\, \partial_t R_k F = W\phi R_k F + \phi R_k F\Omega.$$

Further calculation, due to (10), yields

$$\begin{aligned}\partial_t(\phi R_k F) &= \phi R_k F\Omega C_k(1 - R_k FC_k)^{-1} R_k F - \phi\,\Omega(1 - R_k FC_k)^{-1} R_k F + \phi R_k F\Omega \\ &= \phi R_k F\Omega(1 - C_k R_k F)^{-1} - \phi\,\Omega R_k F(1 - C_k R_k F)^{-1}.\end{aligned}$$

Hence,

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \phi R_k F\Omega(1 - C_k R_k F)^{-1}, 1\rangle - \langle \phi\,\Omega R_k F(1 - C_k R_k F)^{-1}\rangle].$$

Taking into account properties (6) of the pairing, we get

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \phi R_k F\Omega,\ 1\cdot(1 + \hat{R}_k FC_k)^{-1}\rangle - \langle \phi\,\Omega,\ 1\cdot(1 + \hat{R}_k FC_k)^{-1}\hat{R}_k F\rangle]. \tag{12}$$

Now we introduce a dual function $\phi^*(k)$ by means of the relation

$$\tilde{\phi}^*(k) = 1\cdot(1 + \hat{R}_k FC_k)^{-1}. \tag{13}$$

The $\bar{\partial}$-problem for the dual function has the form

$$\bar{\partial}\phi^*(k) = -\iint \mathrm{d}l \wedge \mathrm{d}\bar{l}\ R(l, k)\phi^*(l) \qquad \bar{\partial}\,\tilde{\phi}^*(k) = -\tilde{\phi}^*(k)\hat{R}_k F \tag{14}$$

and $\phi^*(k)$ satisfies the dual spectral problem

$$\partial_x\phi^* + \partial_y\phi^*\sigma_3 - \phi^* Q - \mathrm{i}k[\sigma_3, \phi^*] = 0. \tag{15}$$

The derivation of (14) and (15) is given in appendix C. It is seen from (15) that only the dual function ϕ^* is a true (2+1)-dimensional generalization of inverse functions in 1+1 dimensions. It should be stressed that the definition (13) of the dual function arises naturally within the framework of the formalism based on the representation (5).

Taking into account the above relations concerning the dual function, we write the evolution (12) in the form

$$\begin{aligned}\partial_t Q &= -\mathrm{i}[\sigma_3, \langle \phi R_k F\Omega,\ \tilde{\phi}^*\rangle - \langle \phi\,\Omega,\ \tilde{\phi}^*\hat{R}_k F\rangle] \\ &= -\mathrm{i}[\sigma_3, \langle \bar{\partial}\phi\,\Omega\,\phi^*\rangle + \langle \phi\,\Omega\,\bar{\partial}\phi^*\rangle].\end{aligned}$$

Finally, dividing the dispersion relation into regular and singular parts, we obtain under condition $\Omega_s(k) \to 0$ for $k \to \infty$:

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \bar{\partial}(\phi\Omega_p\phi^*)\rangle - \langle \phi\,\bar{\partial}\,\Omega_s\,\phi^*\rangle]. \tag{16}$$

Assume further that

$$\Omega_p(k) = \alpha_n k^n \sigma_3 \qquad \alpha_n = \text{constant} \quad n = 0, 1, \ldots$$

$$\Omega_s(k) = \frac{1}{2\pi i} \iint \frac{dl \wedge d\bar{l}}{l-k} g(l)\sigma_3$$

which yields $\bar{\partial}\, \Omega_s(k) = g(k)\sigma_3$. Now we introduce a bilocal object

$$M_{12}(x, y_1, y_2, k) = \phi(x, y_1, k)\sigma_3 \phi^*(x, y_2, k) \equiv \phi_1\sigma_3\phi_2^* .$$

It is easy to see that the function M_{12} satisfies the equation

$$\partial_x M_{12} + \sigma_3\partial_{y_1} M_{12} + \partial_{y_2} M_{12}\sigma_3 - ik[\sigma_3, M_{12}] + Q_1 M_{12} - M_{12} Q_2 = 0 \quad (17)$$

where $Q_i \equiv Q(x, y_i)$, $i = 1, 2$. Hence, equation (16) takes the form

$$\delta_{12}\, \partial_t Q_2 = -i\alpha_n \delta_{12}[\sigma_3, \langle\bar{\partial}\, (k^n M_{12})\rangle] + i\delta_{12}[\sigma_3, \langle g(k) M_{12}\rangle] \quad (18)$$

where $\delta_{12} = \delta(y_1 - y_2)$. Following [10], we introduce the notations

$$P_{12}M_{12} = \partial_x M_{12} + \sigma_3\partial_{y_1} M_{12} + \partial_{y_2} M_{12}\sigma_3 \qquad Q_{12}^{\pm} M_{12} = Q_1 M_{12} \pm M_{12} Q_2 . \quad (19)$$

Let M_{12}^{d} and M_{12}^{a} be the diagonal and off-diagonal parts of the matrix M_{12}, respectively. Then (17) and (19) yield

$$P_{12}\, M_{12}^{d} + Q_{12}^{-}\, M_{12}^{a} = 0 \quad (20)$$

$$P_{12}\, M_{12}^{a} - 2ik\sigma_3\, M_{12}^{a} + Q_{12}\, M_{12}^{d} = 0 . \quad (21)$$

We can write from (20) the diagonal part as $M_{12}^{d} = \sigma_3 - P_{12}^{-1}\, Q_{12}^{-} M_{12}^{a}$. Hence, equation (21) is written in the form $(\Lambda - k)M_{12}^{a} = (2i)^{-1}\, Q_{12}^{+} \cdot 1$, where the operator Λ is defined as

$$\Lambda = \frac{1}{2i}\sigma_3(P_{12} - Q_{12}^{-}\, P_{12}^{-1}\, Q_{12}^{-}) .$$

Then $M_{12}^{a} = (2i)^{-1}(\Lambda - k)^{-1}\, Q_{12}^{+} \cdot 1$ and after the expansion $(\Lambda - k)^{-1} = -\sum_{m=1}^{\infty} k^{-m}\Lambda^{m-1}$ we can write the polynomial contribution to $\partial_t Q$ in (18) as

$$-i\alpha_n\delta_{12}[\sigma_3, \langle\bar{\partial}\, (k^n M_{12})\rangle] = \alpha_n\sigma_3\, \delta_{12} \sum_{m=1}^{\infty} \langle\bar{\partial} k^{n-m}\, \Lambda^{m-1}\, Q_{12}^{+} \cdot 1$$

$$= -\frac{i}{2}\alpha_n\sigma_3\, \delta_{12}\, \Lambda^n\, Q_{12}^{+} \cdot 1 .$$

Now we have all we need to formulate a closed system of equations describing the evolution of the potential Q under condition of the linear evolution of the spectral transform R:

$$\begin{aligned} &\delta_{12}\, \partial_t Q_2 = -\frac{i}{2}\alpha_n\sigma_3\, \delta_{12}\, \Lambda^n\, Q_{12}^{+} \cdots 1 + i\delta_{12}[\sigma_3, \langle g(k) M_{12}\rangle] \\ &\left(P_{12}\, M_{12} - ik[\sigma_3, M_{12}] + Q_{12}^{-}\, M_{12}\right) g(k) = 0 . \end{aligned} \quad (22)$$

Here the operator Λ plays the role of a recursion operator (more precisely, Λ is connected with the true recursion operator by means of σ_3 [10]). If $M_{12} = \sigma_3$ and $g(k) = 0$, we get from (22) the well known hierarchy including the Davey–Stewartson-1 equation derived by Santini and Fokas [10] on the basis of an integral representation for W.

For $\Omega_p = 0$ the system (22) takes the form

$$\begin{aligned} &\delta_{12}\,\partial_t Q_2 = \mathrm{i}\delta_{12}[\sigma_3, \langle g(k) M_{12}\rangle] \\ &\left(P_{12} M_{12} - \mathrm{i}k[\sigma_3, M_{12}] + Q_{12}^{-} M_{12}\right) g(k) = 0\,. \end{aligned} \tag{23}$$

It is seen that the structure of the system (23) is similar to that for (1+1)-dimensional Maxwell–Bloch equations ($g(k) \sim \delta(\mathrm{Im}k)\delta(\mathrm{Re}k - \alpha)$):

$$\begin{aligned} &\partial_t Q = \mathrm{i}[\sigma_3, \langle g(k)\Phi(k)\rangle] \qquad \Phi = \phi\,\sigma_3\,\phi^{-1} \\ &\partial_x \Phi(\alpha) - \mathrm{i}\alpha[\sigma_3, \Phi(\alpha)] + [Q, \Phi(\alpha)] = 0 \end{aligned} \tag{24}$$

and the system (23) is reduced to (24) in the (1+1)-dimensional limit. It should be stressed that, as distinct from [3], the (2+1)-dimensional counterpart (23) of the Maxwell–Bloch equations demonstrates explicitly the presence of the 'squared function' term. It can be shown that a function Γ introduced in [3] is expressed, as a matter of fact, in terms of the above squared functions as $\Gamma = \mathrm{i}\bar{\partial}\,(k\phi\,\phi^*)$. Finally, for $n = 2$, equations (22) yield a (2+1)-dimensional generalization of the equations derived in [12] in the context of nonlinear optics.

4. Conclusion

We proposed a procedure for obtaining (2+1)-dimensional nonlinear equations with non-analytic dispersion relations compatible with the linear evolution of the spectral transform. An important step in deriving these equations was to use the representation (5) of the $\bar{\partial}$-problem solution. In spite of the formality of this representation, it allows us to perform all the needed manipulations. The introduction of the dual functions has made it possible to obtain the hierarchy of equations without explicit use of the second Lax operator. The application of the bilocal formalism was crucial for bringing these equations to a form similar to that for their counterparts in 1+1 dimensions.

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Appendix A. Linear spectral problem

We show here that the choice (7) of the dependence of $R(k)$ on spatial variables x and y leads to the Zakharov–Shabat problem on the plane. Differentiating (5) with respect to x

we obtain $\partial_x\phi = \phi\,\partial_x R_k\, FC_k(1 - R_k\, FC_k)^{-1}$. In virtue of the definitions of the integral operators F and C_k and (7) we can perform the following calculation:

$$\begin{aligned}
\phi\,\partial_x R_k\, FC_k &= \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}\iint \mathrm{d}m\wedge\mathrm{d}\bar{m}\ \phi(m)\partial_x R(l,m) \\
&= \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}\iint \mathrm{d}m\wedge\mathrm{d}\bar{m}\ \mathrm{i}m\,\phi(m)\sigma_3\, R(l,m) \\
&\quad - \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}\,\mathrm{i}l \iint \mathrm{d}m\wedge\mathrm{d}\bar{m}\ \phi(m)\, R(l,m)\sigma_3 \\
&= \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}l\wedge\mathrm{d}\bar{l}}{l-k}(\mathrm{i}l\,\phi\sigma_3\, R_l\, F) - \frac{1}{2\pi\mathrm{i}}\iint \mathrm{d}l\wedge\mathrm{d}\bar{l}\ \mathrm{i}\left(1+\frac{k}{l-k}\right)(\phi\, R_l\, F)\sigma_3\,.
\end{aligned} \tag{A.1}$$

Since we have, from (4), $\phi R_k\, FC_k = \phi - 1$, then (A.1) and the evident relation $R_k\, FC_k(1 - R_k\, FC_k)^{-1} = (1 - R_k\, FC_k)^{-1} - 1$ yield

$$\partial_x\phi = -\mathrm{i}k\,\phi\sigma_3 - \mathrm{i}\langle\phi\, R_k\, F\rangle\sigma_3\,\phi + \mathrm{i}k\,\sigma_3(1 - R_k\, FC_k)^{-1}\,. \tag{A.2}$$

Similarly,

$$\partial_y\phi = \mathrm{i}k\,\phi + \mathrm{i}\langle\phi R_k\, F\rangle\phi - \mathrm{i}k\,(1 - R_k\, FC_k)^{-1}\,. \tag{A.3}$$

Adding (A.2) and (A.3) yields

$$\partial_x\phi + \sigma_3\,\partial_y\phi - \mathrm{i}k\,[\sigma_3,\phi] - \mathrm{i}[\sigma_3,\langle\phi R_k\, F\rangle]\phi = 0\,. \tag{A.4}$$

Hence, if we identify $-\mathrm{i}[\sigma_3,\langle\phi R_k\, F\rangle] \equiv Q(x,\ y)$, (A.4) gives the above spectral problem.

Appendix B. Linear evolution problem

In order to derive the linear evolution problem $\partial_t\phi = W\phi + \phi\,\Omega$, we calculate $\partial_t\phi$ from (2) and (5). Let us take for simplicity $\Omega_{\mathrm{p}} = \mathbb{C}$, whereas

$$\Omega(k) = \Omega_{\mathrm{s}}(k) = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}s\wedge\mathrm{d}\bar{s}}{s-k}g(s)\sigma_3$$

which gives $\bar{\partial}\,\Omega(k) = g(k)\sigma_3$. The calculation yields

$$\begin{aligned}
\partial_t\phi &= \phi\,\partial_t R_k\, FC_k(1 - R_k\, FC_k)^{-1} \\
&= [\phi R_k\, F\,\Omega\, C_k - \phi\,\Omega\, R_k\, FC_k](1 - R_k\, FC_k)^{-1} \\
&= \phi R_k\, F\,\Omega\, C_k(1 - R_k\, FC_k)^{-1} - \phi\,\Omega(1 - R_k\, FC_k)^{-1} + \phi\,\Omega \\
&\equiv W(k)\phi(k) + \phi(k)\,\Omega(k)
\end{aligned}$$

where

$$W(k)\phi(k) = (\phi R_k F \Omega C_k - \phi \Omega)(1 - R_k F C_k)^{-1}. \tag{B.1}$$

Taking into account the definitions of the integral operators F and C_k, we can rewrite (B.1) as

$$\begin{aligned} W(k)\phi(k)(1 - R_k F C_k) &= \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}l \wedge \mathrm{d}\bar{l}}{l-k} (\phi(l) R_l F) \Omega(l) - \phi \Omega \\ &= \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}l \wedge \mathrm{d}\bar{l}}{l-k} \iint \mathrm{d}m \wedge \mathrm{d}\bar{m}\ \phi(m) R(l,m) \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}s \wedge \mathrm{d}\bar{s}}{s-l} g(s)\sigma_3 - \phi \Omega. \end{aligned} \tag{B.2}$$

The denominator in (B.2) can be represented as

$$\frac{1}{(l-k)(s-l)} = \frac{1}{s-k}\left(\frac{1}{l-k} - \frac{1}{l-s}\right).$$

Then we have

$$\begin{aligned} W\phi\,(1 - R_k F C_k) &= \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}l \wedge \mathrm{d}\bar{l}}{l-k} \iint \mathrm{d}m \wedge \mathrm{d}\bar{m}\ \phi(m) R(l,m) \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}s \wedge \mathrm{d}\bar{s}}{s-k} g(s)\sigma_3 \\ &\quad - \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}s \wedge \mathrm{d}\bar{s}}{s-k} g(s) \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}l \wedge \mathrm{d}\bar{l}}{l-s} \iint \mathrm{d}m \wedge \mathrm{d}\bar{m}\ \phi(m) R(l,m)\sigma_3 - \phi \Omega \\ &= \phi R_k F C_k \Omega(k) - \frac{1}{2\pi \mathrm{i}} \iint \frac{\mathrm{d}s \wedge \mathrm{d}\bar{s}}{s-k} g(s)(\phi R_s F C_s)\sigma_3 - \phi \Omega \\ &= -\, g(k)\phi(k)\sigma_3 C_k \end{aligned}$$

where, as in appendix A, we use $\phi R_s F C_s = \phi(s) - 1$. Hence,

$$W(k)\phi(k) = -g(k)\phi(k)\sigma_3 C_k (1 - R_k F C_k)^{-1}.$$

It should be noted that the calculation of $\partial_t \phi$ for $\Omega_s = 0$ and $\Omega_p = \alpha_2 k^2 \sigma_3$ on the basis of (5) leads to the well known operator $T_2 = \partial_t - W$ for the Davey–Stewartson-1 equation [5, 12], where a potential A of the mean flow has the form

$$A = 2\sigma_3\big[\langle k(\phi R_k F)\rangle - \langle k\phi R_k F\rangle - \mathrm{i}\langle \partial_y \phi R_k F\rangle\big]^{\mathrm{d}}.$$

Appendix C. Dual spectral problem

Here we give a derivation of (14) and (15). The definition (13) gives $\tilde{\phi}^* = 1 - \tilde{\phi}^* \hat{R}_k F C_k$ and taking into account the evident property $\bar{\partial} f(k) C_k = f(k)$ for any function $f(k)$, this yields

$$\begin{aligned} \bar{\partial}\tilde{\phi}^* = -\tilde{\phi}^* \hat{R}_k F &= -\iint \mathrm{d}l \wedge \mathrm{d}\bar{l}\ \tilde{\phi}^*(l) \hat{R}(k,l) \\ &= -\iint \mathrm{d}l \wedge \mathrm{d}\bar{l}\ \tilde{\phi}^*(l) \tilde{R}(l,k) = -\left[\iint \mathrm{d}l \wedge \mathrm{d}\bar{l}\ R(l,k)\phi^*(l)\right]_{(\mathrm{transpose})}. \end{aligned}$$

Hence, equation (14) follows. Now we find a spectral problem for the dual function ϕ^*. Differentiating (13) with respect to x, we find $\partial_x\tilde{\phi}^* = -\tilde{\phi}^*\partial_x\hat{R}_k\, FC_k(1+\hat{R}_k\, FC_k)^{-1}$. Taking into account $\hat{R}(k,l) = \tilde{R}(l,k)$, we obtain from (7)

$$\partial_x\hat{R}(k,l) = \mathrm{i}k\,\hat{R}(k,l)\sigma_3 - \mathrm{i}l\,\sigma_3\hat{R}(k,l) \qquad \partial_y\hat{R}(k,l) = -\mathrm{i}(k-l)\hat{R}(k,l).$$

Then following the calculation in appendix A, we obtain

$$\partial_x\tilde{\phi}^* = \mathrm{i}\tilde{\phi}^*\sigma_3 - \mathrm{i}\langle\tilde{\phi}^*\hat{R}_k\,F, 1\rangle\sigma_3\tilde{\phi}^* - \mathrm{i}k\,\sigma_3(1+\hat{R}_k\,FC_k)^{-1}$$

and

$$\partial_x\phi^* = \mathrm{i}k\,\sigma_3\phi^* - \mathrm{i}\phi^*\sigma_3\langle 1, \tilde{\phi}^*\hat{R}_k\,F\rangle - \mathrm{i}k(1+\hat{R}_k\,FC_k)^{-1}_{(\text{transpose})}\sigma_3\,.$$

Similarly,

$$\partial_y\phi^* = -\mathrm{i}k\,\phi^* + \mathrm{i}\phi^*\langle 1, \tilde{\phi}^*\hat{R}_k\,F\rangle + \mathrm{i}k(1+\hat{R}_k\,FC_k)^{-1}_{(\text{transpose})}.$$

Hence,

$$\partial_x\phi^* + \partial_y\phi^*\sigma_3 - \mathrm{i}k[\sigma_3, \phi^*] + \mathrm{i}\phi^*[\sigma_3, \langle 1, \tilde{\phi}^*\hat{R}_k\,F\rangle] = 0\,. \qquad \text{(C.1)}$$

Now we need a connection of $\langle 1, \tilde{\phi}^*\hat{R}_k\,F\rangle$ with Q. It can be found as follows:

$$\begin{aligned}\langle\phi R_k\,F, 1\rangle &= \langle 1\cdot(1-R_k\,FC_k)^{-1}R_k\,F, 1\rangle = \langle 1\cdot(1-R_k\,FC_k)^{-1}, \hat{R}_k\,F\rangle \\ &= \langle 1, \hat{R}_k\,F(1+C_k\hat{R}_k\,F)^{-1}\rangle = \langle 1,\ 1\cdot(1+\hat{R}_k\,FC_k)^{-1}\hat{R}_k\,F\rangle \\ &= \langle 1, \tilde{\phi}^*\hat{R}_k\,F\rangle\,.\end{aligned}$$

Hence,

$$Q = -\mathrm{i}[\sigma_3, \langle\phi R_k\,F, 1\rangle] = -\mathrm{i}[\sigma_3, \langle 1, \tilde{\phi}^*\hat{R}_k\,F\rangle]$$

and we derive from (C.1) the equation (15).

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